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Frequency domain analysis for suppression of output vibration from periodic disturbance using nonlinearities

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Abstract

A frequency domain approach is proposed to suppress output vibration from periodic exogenous disturbances for SISO systems by using nonlinear feedback. Based on the frequency domain theory of nonlinear Volterra systems, the analytical relationship between system output frequency response and controller parameters is obtained, and a series of associated theoretical results and techniques are discussed for the purpose of nonlinear feedback analysis and design. It is shown that a low degree nonlinear feedback may be sufficient for some control problems. A general procedure is provided for this frequency domain analysis and design. This paper provides a systematic frequency domain approach to exploiting the potential advantage of nonlinearities to achieve a desired frequency domain performance for active/passive vibration control or energy dissipation systems. The new approach is demonstrated through an analysis and design of a nonlinear feedback for a simple vibration control system. By properly introducing a simple nonlinear damping to the system, the performance of the system output response when subject to a periodic disturbance is improved, compared with a linear damping controller.

1. Introduction

Suppression of periodic disturbances covers a wide range of applications, for example, active control and isolation of vibrations in engineering and vehicle systems. Traditionally, an increase in damping can reduce the response at the resonance. However, this is often at the expense of degradation of isolation at high frequencies [1]. Many methods have been proposed to deal with this problem, such as optimal control, H-infinity control, "skyhook" damper, repetitive learning control, optimization, etc. [1–4]. A much more comprehensive and up-to-date survey can refer to Ref. [5]. Nonlinear feedback is an approach that has been noted recently by some researchers [6–8]. It is shown in Ref. [4] that, although it is not possible to use linear time-invariant controllers to robustly stabilize a controlled plant and to achieve asymptotic rejection of a periodic disturbance, the problem is solvable by using a nonlinear controller for a linear plant subjected to a triangular

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wave disturbance. Based on the Hamiltonian system theory, an optimal nonlinear feedback control strategy is proposed in Ref. [8] for randomly excited structural systems. It has also been reported many times that existing nonlinearities or deliberately introduced nonlinearities may bring benefits to control system design [1]. Hence, the design of a nonlinear feedback controller to suppress periodic disturbances has great potential to achieve a considerably improved control performance. However, it should be noted that most of these existing methods mentioned above are based on state space and in the time domain, and some of those usually involve complicated design procedure in mathematics.

Recently, some progress has been achieved in the analysis of nonlinear systems in the frequency domain [9–13]. The algorithms for determination of the generalized frequency-response functions (GFRFs) and output frequency response have been obtained for nonlinear Volterra systems [9]. Based on these results, the concept of output frequency-response function (OFRF) for nonlinear Volterra systems was proposed in Ref. [13], and was further studied in Ref. [10]. The OFRF reveals an analytical relationship between system output spectrum and system model parameters for a wide class of nonlinear systems and provides an important basis for the analysis and design of output behaviour of nonlinear systems in the frequency domain. For a linear controlled plant subject to periodic disturbances, if a nonlinear feedback is introduced to produce a nonlinear closed-loop system, the relationship between the disturbance and the system output is nonlinear and can, under certain conditions, be described in the frequency domain by using the OFRF to explicitly relate the controller parameters to the system output frequency response. Therefore, by properly designing the controller parameters based on this explicit relationship defined by the OFRF, the effect of the periodic disturbance on the system output frequency response could be significantly suppressed. Motivated by this idea, a frequency domain approach to analysis and design of nonlinear feedback for the exploitation of the potential advantage of nonlinearities is proposed in this study to suppress sinusoidal exogenous disturbances for a linear controlled plant.

This paper is organized as follows. The problem formulation is given in Section 2, which is divided into several basic problems that can be addressed separately. Section 3 is concerned with some fundamental issues of the analysis and design of nonlinear feedback corresponding to different basic problems. Some theoretical results and techniques needed to solve these basic problems are established. Section 4 illustrates the implementation of the new approach by tackling a simple vibration system. Simulation results are provided to demonstrate the new approach.

2. Problem formulation

Consider an SISO linear system described by the following differential equation:

$$\sum_{l=0}^{L} C_x(l) D^l x + bu + e\eta = 0, \tag{1}$$

$$y = \sum_{l=0}^{L-1} C_y(l) D^l x + \mathrm{d}u, \tag{2}$$

where x, y, u, $\eta \in \Re^1$ represent the system state, output, control input, and an exogenous disturbance input, respectively; η stands for a known, external, bounded and periodical vibration, which can be described by a summation of multiple sinusoidal functions; L is a positive integer; D^l is an operator defined by $D^l x = \mathrm{d}^l x/\mathrm{d}t^l$. The model of system (1–2) can also be written into a state-space form

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u + \mathbf{E}\eta, \tag{3}$$

$$y = \mathbf{CX} + \mathrm{d}u, \tag{4}$$

where $\mathbf{X} = [x,\ D^1 x, \ldots, D^{L-1} x]^{\mathrm{T}} \in \Re^L$ is the system state variable, $\mathbf{A}$ and $\mathbf{C}$ are matrixes with appropriate dimensions, $\mathbf{B} = [0_{1\times(L-1)},\ b]^{\mathrm{T}}$, $\mathbf{E} = [0_{1\times(L-1)},\ e]^{\mathrm{T}}$. The problem to be addressed in the present study is as follows.

Given a frequency interval $I(\omega)$ and a desired magnitude level of the output frequency response Y^* over this frequency interval, find a nonlinear state feedback control law

$$u = -\varphi(x, D^1x, \ldots, D^{L-1}x) \tag{5}$$

such that

$$\max_{\omega \in I(\omega)} (Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega)) \leqslant Y^*, \tag{6a}$$

where the feedback control law $-\varphi(x, D^1x, \ldots, D^{L-1}x)$ is generally a nonlinear function of $x, D^1x, \ldots, D^{L-1}x$, with the linear state/output feedback as a special case; $Y(\mathrm{j}\omega)$ is the spectrum of the system output. In order to achieve Eq. (6a), at present it can be realized by transforming Eq. (6a) to

$$\max_{\substack{\omega_k \in I(\omega) \\ k=1,2,\ldots,\bar{k}}} (Y(\mathrm{j}\omega_k)Y(-\mathrm{j}\omega_k)) \leqslant Y^*. \tag{6b}$$

That is, evaluate the output spectrum at a series of frequency points such that the maximum value is suppressed to a desired level. The desired control law (5) should satisfy Eq. (6b). In the following, suppose $I(\omega) = \omega_0$, that is only the output response at a specific frequency is considered. Let $Y = Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega)\big|_{(\omega_0,u)}$, then $Y_0 = Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega)\big|_{(\omega_0,0)}$ shows the magnitude of the system output frequency response at frequency ω_0 under zero control input. Obviously, it should be

$$Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega)\big|_{(\omega_0,u)} \leqslant Y^* < Y_0 = Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega)\big|_{(\omega_0,0)}. \tag{7}$$

To obtain a nonlinear feedback controller, $\varphi(x, D^1x, \ldots, D^{L-1}x)$ can be written into a polynomial form in terms of $x, D^1x, \ldots, D^{L-1}x$ as

$$\varphi(x, D^1x, \ldots, D^{L-1}x) = \sum_{p=1}^{M} \sum_{l_1,\cdots,l_p=0}^{L-1} C_{p0}(l_1, \ldots, l_p) \prod_{i=1}^{p} D^{l_i}x, \tag{8}$$

where M is a positive integer representing the maximum degree of nonlinearity in terms of $D^ix(t)$ $(i = 0, \ldots, L-1)$; $\sum_{l_1\cdots l_p=0}^{L-1}(\cdot) = \sum_{l_1=0}^{L-1}\cdots\sum_{l_p=0}^{L-1}(\cdot)$. The nonlinear function in Eq. (8) includes a general class of possible linear and nonlinear functions of D^ix $(i = 0,\ldots,L-1)$, which also enables the frequency-response functions of the closed-loop system can be obtained by the existing theory of the authors in Refs. [9–11]. Since $D^ix = e(i+1)^{\mathrm{T}}\mathbf{X}$, where $e(i+1)$ is an L-dimensional column vector whose $(i+1)$th element is 1 with all other terms zero, $\varphi(x, D^1x, \ldots, D^{L-1}x)$ can also be written as a function of $\mathbf{X}$, i.e., $\varphi(\mathbf{X})$. For the parameters $C_{p0}(.)$ $(p = 1, \ldots, M)$, when $p = 1$ the parameters will be referred to as the linear parameters corresponding to the linear terms in Eq. (8), e.g., $C_{1,0}(2)(\mathrm{d}^2x(t)/\mathrm{d}t^2)$. All the other parameters in Eq. (8) will be referred to as the nonlinear parameters corresponding to the nonlinear terms $\prod_{i=1}^{p} D^{l_i}x(t)$. p is the nonlinear degree of the nonlinear parameter $C_{p0}(\cdot)$. Let

$$C(M, L) = \left(C_{p0}(l_1, \ldots, l_p) \,\middle|\, \begin{array}{c} p = 1, \ldots, M \\ l_i = 0, \ldots, L-1 \\ i = 1, \ldots, p \end{array} \right), \tag{9}$$

which include all the parameters in Eq. (8). Substituting Eq. (8) into Eqs. (1) and (2) yields the closed-loop system as

$$\sum_{p=1}^{M} \sum_{l_1,\ldots,l_p=0}^{L} \bar{C}_{p0}(l_1, \ldots, l_p) \prod_{i=1}^{p} D^{l_i}x + e\eta = 0, \tag{10a}$$

$$\sum_{p=1}^{M} \sum_{l_1,\ldots,l_p=0}^{L-1} \tilde{C}_{p0}(l_1, \ldots, l_p) \prod_{i=1}^{p} D^{l_i}x = y, \tag{10b}$$

where

$$\bar{C}_{10}(l_1) = C_x(l_1) - bC_{10}(l_1), \quad \tilde{C}_{10}(l_1) = C_y(l_1) - \mathrm{d}C_{10}(l_1),$$

$$\bar{C}_{p0}(l_1, \ldots, l_p) = -bC_{p0}(l_1, \ldots, l_p), \tilde{C}_{p0}(l_1, \ldots, l_p) = -\mathrm{d}C_{p0}(l_1, \ldots, l_p)$$

for $p = 1, \cdots, M$, $l_i = 0, \cdots, L$, and $i = 1, \cdots, p$. System (10) is described by a nonlinear differential equation model, whose GFRF can be obtained [9]. According to the results in Ref. [14], the model can represent a wide class of nonlinear systems. This implies that the nonlinear control law (8) can be used to achieve different control objectives of interests.

The task for the nonlinear feedback design is to determine M and a range for the controller parameters in Eq. (9) to make the closed-loop system (10) bounded stable around its zero equilibrium, and then to determine the specific values for the controller parameters from the OFRF which defines the relationship between the closed-loop system output spectrum and controller parameters to achieve the required steady-state performance (7).

There are generally four fundamental issues to be addressed for the nonlinear feedback design problem as follows:

(a) Determination of the analytical relationship between the system output spectrum and the nonlinear controller parameters.
(b) Determination of an appropriate structure for the nonlinear feedback controller. Only significant nonlinear terms are needed in the controller to achieve the control objective.
(c) Derivation of a range for the values of the control parameters over which the stability of the closed-loop nonlinear system is guaranteed.
(d) Development of an effective numerical method for the practical implementation of the feedback controller design.

The focus of Section 3 is to investigate these fundamental issues. A simulation study will be presented thereafter to illustrate these results.

3. Fundamental results for nonlinear feedback analysis and design

3.1. OFRF

In this section, the output frequency response of the closed-loop nonlinear system (10) is derived. The relationship between the system output spectrum and the controller parameters are investigated to produce an important basis for the nonlinear feedback analysis and design.

3.1.1. Output spectrum of the closed-loop system

The nonlinear frequency domain approach in this study is based on the Volterra-series approximation. It has been shown that, any time invariant, causal, nonlinear system with fading memory can be approximated by a finite Volterra series [15]. It was also proved in Refs. [16,17] the existence of a locally convergent Volterra-series representation for the input–output relation of a large class of continuous-time nonlinear systems. Therefore, with the BIBO stability condition for the controller parameters which will be studied in Section 3.3, the relationship between the output $y(t)$ and the input $\eta(t)$ of system (10) can be approximated by Volterra functional polynomials up to the Nth order as

$$y(t) = \sum_{n=1}^{N} y_n(t), \quad y_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} \eta(t - \tau_i)\, \mathrm{d}\tau_i, \tag{11}$$

where $h_n(\tau_1, \ldots, \tau_n)$ is the nth-order Volterra kernel of system (10) corresponding to the input–output relationship from $\eta(t)$ to $y(t)$. When Eq. (11) is a multitone function

$$\eta(t) = \sum_{i=1}^{K} |F_i| \cos(\omega_i t + \angle F_i) \tag{12}$$

(note that F_i is a complex number, $\angle F_i$ is the argument and $|F_i|$ is the modulus) the system output spectrum can be obtained by extending the result in [11]

$$Y(\mathrm{j}\omega) = \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n}) F(\omega_{k_1})\cdots F(\omega_{k_n}), \tag{13}$$

where

$$F(\omega) = \begin{cases} |F_i|\mathrm{e}^{\mathrm{j}\angle F_i} & \text{if } \omega \in \{\omega_k, k=\pm 1,\ldots,\pm K\}, \\ 0 & \text{else}, \end{cases} \tag{14}$$

$$H_n(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n)\mathrm{e}^{-\mathrm{j}(\omega_1\tau_1+\cdots+\omega_n\tau_n)}\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n. \tag{15}$$

Eq. (15) is the nth-order GFRF of system (10) for the relationship between $\eta(t)$ and $y(t)$, which can be obtained as follows.

Proposition 1. *The GFRFs $H_n(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n})$ from the disturbance $\eta(t)$ to the output $y(t)$ of nonlinear system* (10) *can be determined as*

$$H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = \sum_{p=1}^{n}\sum_{l_1\cdots l_p=0}^{L-1} \tilde{C}_{p0}(l_1,\ldots,l_p) H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n), \tag{16a}$$

$$H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = \sum_{i=1}^{n-p+1} H_i^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_i) H_{n-i,p-1}^1(\mathrm{j}\omega_{i+1},\ldots,\mathrm{j}\omega_n)(\mathrm{j}\omega_1+\cdots+\mathrm{j}\omega_i)^{l_p}, \tag{16b}$$

$$H_{n,1}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)(\mathrm{j}\omega_1+\cdots+\mathrm{j}\omega_n)^{l_1},\;\; H_1^1(\mathrm{j}\omega_1) = e \Big/ \sum_{l_1=0}^{L} \bar{C}_{10}(l_1)(\mathrm{j}\omega_1)^{l_1}, \tag{16c}$$

$$H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = -\frac{1}{e} H_1^1(\mathrm{j}\omega_1+\cdots+\mathrm{j}\omega_n)\left(\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L-1} \bar{C}_{p0}(l_1\cdots l_p) H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) - e\delta(n-1)\right) \tag{16d}$$

and

$$\delta(n) = \begin{cases} 1 & n=0, \\ 0 & \text{otherwise}. \end{cases}$$

See Appendix A for the proof. Note that the nth-order GFRF from $\eta(t)$ and $x(t)$ can directly be obtained from Ref. [9] which is denoted by $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$. However, the nth-order GFRF from $\eta(t)$ and $y(t)$ cannot be computed by directly applying the results in Ref. [9], because system (10) having a nonlinear output is not consistent with the model studied in Refs. [9,10,13]. From Proposition 1, the GFRFs can be computed recursively from the time domain model (10), and the output spectrum of system (10) can be obtained analytically from Eqs. (13,16), which are an explicit function of the parameters in the control law (8). Therefore, the control law (8) can be studied in the frequency domain. In order to make clear the analytical relationship between the system output spectrum and model parameters from these recursive computations and to make light on the selection of feedback nonlinearities which are to be possibly included in the control law (8), the OFRF of system (10) can be expressed as a polynomial function of the nonlinear controller parameters in Eq. (9) according to Refs. [10,13], i.e.,

$$Y(\mathrm{j}\omega) = P_0(\mathrm{j}\omega) + a_1 P_1(\mathrm{j}\omega) + a_2 P_2(\mathrm{j}\omega) + \cdots, \tag{17a}$$

where $P_0(\mathrm{j}\omega)$ is the linear part of the system output frequency response, $P_i(\mathrm{j}\omega)$ ($i \geqslant 1$) represents the effects of higher order nonlinearities, and a_i ($i = 1,2,\ldots$) are functions of the nonlinear controller parameters which can be determined by following Ref. [10]. Moreover, for a nonlinear controller parameter c in Eq. (9), there exists a series of functions of frequency ω $\{\bar{P}_i(\mathrm{j}\omega),\ i=0,1,2,3,\ldots\}$ such that

$$Y(\mathrm{j}\omega)=\bar{P}_0(\mathrm{j}\omega)+c\bar{P}_1(j\omega)+c^2\bar{P}_2(\mathrm{j}\omega)+\cdots. \tag{17b}$$

The property above explicitly demonstrates the relationship between the system output spectrum and the nonlinear controller parameters, and enables the OFRF to be determined by using a simple numerical method, which will be discussed in Section 3.4. Thus, it considerably facilitates the analysis and design of the nonlinear feedback in the frequency domain. In order to reveal the contribution of the nonlinear controller parameters of different degrees to the output spectrum clearly and thus make light on the structure selection of the control law (8), some useful results regarding the parametric characteristic of the OFRF are discussed in the following section.

3.1.2. Parametric characteristic analysis of the output spectrum

The parametric characteristic analysis of the system output spectrum is to investigate the polynomial structure of Eq. (17a) in detail, and to reveal how the frequency-response functions in Eqs. (13,16a–d) depend on the nonlinear controller parameters (i.e., $C_{p0}(.)$ for $p>1$) in Eq. (9).

Define the pth-degree parameter vector

$$\mathbf{C}_{p0}=\left[C_{p0}(0,\ldots,0),C_{p0}(0,\ldots,1),\ldots,C_{p0}(\underbrace{L,\ldots,L}_{p})\right],$$

which includes all the parameters of degree p in Eq. (9). To obtain the parametric characteristics of the output spectrum, the coefficient extraction (CE) operator is needed, which has two operations "$\otimes$" and "$\oplus$", and was defined in Ref. [10]. Following the method in Ref. [10], the parametric characteristics of the GFRF $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ from $u(t)$ to $y(t)$ can be obtained as for $n>1$

$$\begin{aligned}\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))&=\bigoplus_{p=2}^{n}(C_{p,0}\otimes\mathrm{CE}(H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)))=\bigoplus_{p=2}^{n}(C_{p,0}\otimes\mathrm{CE}(H_{n-p+1}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)))\\&=C_{n0}\oplus\bigoplus_{p=2}^{[(n+1)/2]}(C_{p0}\otimes\mathrm{CE}(H_{n-p+1}^1(\cdot))).\end{aligned} \tag{18}$$

For $n=1$, $\mathrm{CE}(H_1^1(\mathrm{j}\omega_1))=1$. Here, $[n/2]$ means to get the integer part of [.]. From the invariant property of the CE operator, it follows for the nonlinear controller parameters in Eq. (9) that

$$\mathrm{CE}(\bar{C}_{p0}(l_1,\ldots,l_p))=C_{p0}(l_1,\ldots,l_{p+q}),\mathrm{CE}(\tilde{C}_{p0}(l_1,\ldots,l_p))=C_{p0}(l_1,\ldots,l_p).$$

Applying CE operator to Eq. (16a) for the nonlinear parameters in Eq. (9),

$$\begin{aligned}\mathrm{CE}(H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))&=\mathrm{CE}\left(\sum_{l_1=0}^{L}\tilde{C}_{1,0}(l_1)H_{n,1}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)+\sum_{p=2}^{n}\sum_{l_1,\ldots,l_p=0}^{L}\tilde{C}_{p0}(l_1,\ldots,l_p)H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)\right)\\&=\mathrm{CE}\left(\sum_{l_1=0}^{L}(C_y(l_1)-C_{10}(l_1))H_{n,1}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)\right.\\&\quad\left.+\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}(-d)C_{p0}(l_1,\ldots,l_p)H_{n,p}^1(j\omega_1,\ldots,j\omega_n)\right)\\&=\begin{cases}1, & n=1,\\ \bigoplus_{p=2}^{n}(C_{p0}\otimes\mathrm{CE}(H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))), & n>1.\end{cases}\end{aligned} \tag{19}$$

Therefore, with respect to the nonlinear parameters in Eq. (9), the parametric characteristics of the GFRFs $H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ from $\eta(t)$ to $y(t)$ is the same as those of the GFRFs $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ from $u(t)$ to $y(t)$, i.e.,

$$\mathrm{CE}(H_n^2(\cdot))=\mathrm{CE}(H_n^1(\cdot))\quad\text{for}\ \ n>0. \tag{20}$$

That is, the effect from the nonlinear parameters in Eq. (9) on the GFRFs $H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ is the same as that on the GFRFs $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$. Eqs. (18)–(20) reveal how the GFRFs depend on the nonlinear controller parameters in Eq. (9). Based on these results, the parametric characteristic of the OFRF can be obtained as

$$\begin{aligned}
\mathrm{CE}(Y(\mathrm{j}\omega)) &= \mathrm{CE}\left(\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^2(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n})F(\omega_{k_1}),\ldots,F(\omega_{k_n})\right)\\
&= \mathrm{CE}\left(\sum_{n=1}^{N}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^2(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n})\right) = \mathrm{CE}\left(\sum_{n=1}^{N} H_n^2(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n})\right)\\
&= \mathrm{CE}(H_1^2(\cdot))\oplus\mathrm{CE}(H_2^2(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^2(\cdot))\\
&= \mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot)).
\end{aligned}\tag{21a}$$

Therefore, according to the results in Ref. [10], there exist a complex-valued function vector $\tilde{F}_n(\mathrm{j}\omega)$ with appropriate dimension such that

$$Y(j\omega) = \left(\overset{N}{\underset{n=1}{\oplus}}\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))\right)\tilde{F}_n(\mathrm{j}\omega). \tag{21b}$$

This is the detailed polynomial function of Eq. (17a). Eq. (21b) provides a straightforward expression for the relationship between system output spectrum and the controller parameters. Now the coefficients of the polynomial function (17a) can be determined as

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_K\end{bmatrix} = \mathrm{CE}(Y(\mathrm{j}\omega)) = \mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot)), \tag{21c}$$

where K is the dimension of the vector $\mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot))$.

In order to understand more these parametric characteristics, the following results are given.

Proposition 2. *The elements in* $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$ *include and only include all the parameter monomial* (*consisting of the nonlinear parameters in Eq.* (9)) *in* $\mathbf{C}_{p0}\otimes\mathbf{C}_{r_10}\otimes\mathbf{C}_{r_20}\otimes\cdots\otimes\mathbf{C}_{r_k0}$ *for* $0\leqslant k\leqslant n-2$, *satisfying* $p+\sum_{i=1}^{k}r_i = n+k,\ 2\leqslant r_i\leqslant n-1$, *and* $0\leqslant p\leqslant n$.

Proposition 2 is a simple case of Proposition 2 in Ref. [10], and demonstrates whether and how a nonlinear parameter in Eq. (9) is included in $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$. Different parameters may form one monomial acting as an element in $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$, and thus have a coupled effect on $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$. If a nonlinear parameter appears in $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$, this implies that it has an effect on $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ and thus on $Y(\mathrm{j}\omega)$. If this nonlinear parameter is an independent element in $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$, then it has an independent effect on $Y(\mathrm{j}\omega)$. Furthermore, if a parameter frequently appears in $\mathrm{CE}(H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n))$ with different monomial degrees, this may implies that this parameter has more strong effect on $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ and thus $Y(\mathrm{j}\omega)$. For this reason, the parametric characteristic analysis of $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ can make light on the effect from different nonlinear parameters on $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ and $Y(\mathrm{j}\omega)$.

From Proposition 2, the term $(\mathbf{C}_{n0})^i$ should be included in the GFRF $H_m(.)$, where m is computed as $m+k = m+i-1 = ni$. Hence, $m = ni-i+1 = 1+(n-1)i$. It can be seen that, when n is smaller, $\mathbf{C}_{n,0}$ will contribute independently to more orders of the GFRFs whose orders are $(n-1)i+1$ for $i = 1,2,3,\ldots$; and if n is larger, $\mathbf{C}_{n,0}$ can only affect the GFRFs of order higher than n. It is known that for a Volterra system, the system nonlinear dynamics is usually dominated by the first several order GFRFs [15]. This implies that the nonlinear terms with coefficient $\mathbf{C}_{n,0}$ of smaller nonlinear degree, e.g., 2 and 3, take much greater roles in the GFRFs than other pure output nonlinearities. This property is significant for the selection of possible nonlinear terms in the feedback design. Moreover, it can be verified from Proposition 2 that, if the 2nd and 3rd degree nonlinear control parameters are all zero, i.e., $\mathbf{C}_{20} = 0$ and $\mathbf{C}_{30} = 0$, then $H_2(.) = 0$, and $H_3(.) = 0$. However, even if $\mathbf{C}_{n0} = 0$ (for $n>3$), the nth-order GFRF $H_n(.)$ is not zero, providing there are nonzero terms in $\mathbf{C}_{20}$ or $\mathbf{C}_{30}$. This further demonstrates that the nonlinear controller parameters in $\mathbf{C}_{20}$ and $\mathbf{C}_{30}$ have a more important role in the determination of the GFRFs than any other nonlinear parameters, and thus has a more important effect on the output spectrum. These imply that a lower degree nonlinear feedback may be sufficient for some control problems. These provide a guidance to choose from the nonlinear candidate terms included in Eq. (9).

3.2. The structure of the nonlinear feedback

The determination of the structure for the nonlinear state feedback function (8) is an important task to be tackled. Firstly, as discussed in Section 3.1.2, the structure parameter M in Eq. (8) should be chosen as small as possible since lower degree of nonlinear terms have much larger contributions in the output spectrum. It can be increased gradually until the control objective is achieved. Secondly, after M is determined, whether a term in $\mathbf{C}_{p0}$ is effective or not can be checked. An effective controller must satisfy inequality (7). Thus for the effectiveness of a specific nonlinear controller parameter c, this requirement can be written as

$$\frac{\partial\left|Y(\mathrm{j}\omega_0)\right|}{\partial c}<0 \quad \text{for some } c. \tag{22}$$

Consider the specific nonlinear controller parameter c in $\mathbf{C}_{p0}$ and let all the other nonlinear controller parameters be zero or assumed to be constants. Then only the nonlinear coefficient c^i appears in $\mathrm{CE}(H^1_{1+(p-1)i}(.))$ according to Proposition 2. Therefore, only the GFRFs for the orders $1+(p-1)i$ (for $i=1,2,3,\ldots$) need to be computed to obtain the system output spectrum in Eq. (13). According to Eq. (21), the output spectrum can be written as

$$Y(\mathrm{j}\omega;c)=\bar{P}_0(\mathrm{j}\omega)+c\bar{P}_1(\mathrm{j}\omega)+c^2\bar{P}_2(\mathrm{j}\omega)+\cdots. \tag{23}$$

It can easily be shown that if $\mathrm{Re}(\bar{P}_0(\mathrm{j}\omega)\bar{P}_1(-\mathrm{j}\omega))<0$ then there must exist $\varepsilon>0$ such that $((\partial|Y(\mathrm{j}\omega)|)/\partial c)<0$ for $0<c<\varepsilon$ or $-\varepsilon<c<0$, where $\mathrm{Re}(\cdot)$ is to get the real part of (.). This can be used to find the nonlinear terms, which are effective. For simplicity, $\bar{P}_1(-\mathrm{j}\omega)$ can be computed from Eqs. (12) to (16) by letting the other nonlinear parameters to be zero and $\bar{P}_0(\mathrm{j}\omega)$ is the linear part of the output spectrum in this case. Only the effective nonlinear terms in $C(M)$ is considered. By this way, the structure of the nonlinear function (8) can be determined. It shall be noted that, in this process the output spectrum needs to be analytically computed up to at most the third order by using Eqs. (12)–(16). The structure of the control law (8) can also be determined by simply including all the possible nonlinear terms of degree up to M. Once the output spectrum is determined by the numerical method in Section 3.4, the detailed values of the coefficients of these nonlinear terms can be optimized for the control objective (7) in the stability region developed in the following section. If objective (7) cannot be achieved after M is enough large, this may implies that objective (7) cannot be achieved by control (8) and the best solution by far can be used as the optimal one in this case.

3.3. Stability of the closed-loop system

As mentioned above, the stability of a nonlinear system should be guaranteed such that the nonlinear system can be approximated by a locally convergent Volterra series. Therefore, a range for the nonlinear controller parameters which can ensure the stability of the closed-loop system (10) can be determined. For simplicity, Eq. (10) can also be written into a state-space form as

$$\dot{\mathbf{X}}=\mathbf{AX}-\mathbf{B}\varphi(\mathbf{X})+\mathbf{E}\eta:=f(\mathbf{X})+\mathbf{E}\eta, \tag{24a}$$

$$y=\mathbf{CX}-\mathbf{D}\varphi(\mathbf{X}):=h(\mathbf{X}). \tag{24b}$$

The definitions of **A**, **B**, **C**, **D**, **E** are appropriate matrices which are the same as system (3–4). Noting that the exogenous disturbance in Eq. (24) is a periodic bounded signal, and the objective in vibration control is often to suppress the output vibration below a desired level, a concept of asymptotic stability to a ball is adopted in this section. This concept implies that the magnitude of the output for a system is asymptotically controlled to a satisfactory predefined level. Based on this concept, a general result is then derived to ensure the stability of the closed-loop nonlinear system (24).

A Ball $B_\rho(\mathbf{X})$ is defined as $B_\rho(\mathbf{X}) = \{\mathbf{X} | \|\mathbf{X}\| \leqslant \rho, \rho > 0\}$. A K-function $\gamma(s)$ is an increasing function of s, and a KL-function $\beta(s,t)$ is an increasing function of s, but a decreasing function of t. For detailed definitions of K/KL-functions can refer to Ref. [18].

Asymptotic stability to a Ball: Given an initial state $\mathbf{X}_0 \in \Re^n$ and disturbance input η of a nonlinear system, if there exists a KL-function β such that the solution $\mathbf{X}(t, \mathbf{X}_0, \eta)$ (for $t \geqslant 0$) of the system satisfies $\|\mathbf{X}(t,\mathbf{X}_0,\eta)\| \leqslant \beta(\|\mathbf{X}_0\|, t) + \rho$, $\forall t > 0$, then the system is said to be asymptotically stable to a ball $B_\rho(\mathbf{X})$, where ρ is an upper bound function of η, i.e., there exist a K-function γ such that $\rho = \gamma(\|\eta\|_\infty)$.

Assumption 1. There exists a K-function σ such that the output function $h(\mathbf{X})$ of the nonlinear system (24) satisfies $\|h(\mathbf{X})\| \leqslant \sigma(\|\mathbf{X}\|)$.

Proposition 3. *Suppose Assumption* 1 *holds, then the following statements are equivalent*:

(a) *There exist a smooth function* $V: \Re^L \to \Re_{\geqslant 0}$ *and* K_∞*-functions* β_1, β_2 *and K-functions* α, γ *such that*

$$\beta_1(\|\mathbf{X}\|) \leqslant V(\mathbf{X}) \leqslant \beta_2(\|\mathbf{X}\|) \text{ and } \frac{\partial V(\mathbf{X})}{\partial \mathbf{X}}\{f(\mathbf{X}) + \mathbf{E}\eta\} \leqslant -\alpha(\|\mathbf{X}\|) + \gamma(\|\eta\|_\infty). \tag{25}$$

(b) *System* (24) *is asymptotically stable to the ball* $B_\rho(\mathbf{X})$ *with* $\rho = \beta_1(2\beta_2^{-1}\alpha^{-1}\gamma(\|\eta\|_\infty))$, *and the output of system* (24) *is asymptotically stable to the ball* $B_{\sigma(2\rho)}(y)$.

Proof. See Appendix A. □

Note that Proposition 3 can guarantee the asymptotical stability to a ball of system (24) when subject to bounded disturbance, and asymptotical stability to zero when the disturbance goes to zero. This is right the property of fading memory which is required for the existence of a convergent Volterra-series approximation for the system input–output relationship [15]. Though it is not easy to derive a general stability condition for the general controller (5), there are always various methods [19] to choose a proper Lyapunov function based on Proposition 3 to derive a stability condition for a specific controller.

3.4. A numerical method for the nonlinear feedback controller design

The nonlinear controller parameters can be determined by solving Eq. (17) to satisfy performance (6) or (7) under the stability condition. However, it can be seen that the analytical derivation of the output spectrum of system (10) involves complicated symbolic computation for higher nonlinear orders than 5. To circumvent this problem, as discussed in Section 3.1.1, the following numerical method can be used since the detailed polynomial structure of the OFRF is known by using the method in Section 3.1:

(1) The system OFRF can be expressed as $Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega) = |Y(\mathrm{j}\omega)|^2 = \mathbf{C}\,\tilde{\mathbf{P}}(\mathrm{j}\omega)$ according to Eq. (21) with a finite polynomial degree, where $\tilde{\mathbf{P}}(\mathrm{j}\omega)$ is a complex-valued function vector,

$$\begin{aligned}\mathbf{C} &= \begin{bmatrix} 1 & c_1 & c_2 & c_3 & \cdots & c_{K!} \end{bmatrix} \\ &= (\mathrm{CE}(H_1^1(\cdot)) \oplus \mathrm{CE}(H_2^1(\cdot)) \oplus \cdots \oplus \mathrm{CE}(H_N^1(\cdot))) \otimes (\mathrm{CE}(H_1^1(\cdot)) \oplus \mathrm{CE}(H_2^1(\cdot)) \oplus \cdots \oplus \mathrm{CE}(H_N^1(\cdot))).\end{aligned}$$

(2) Collect the system time domain steady output $y_i(t)$ under different values of the controller parameters $\mathbf{C}_i = [1\ c_{1i}, c_{2i}, \ldots, c_{(K!)i}]$ for $i = 1,2,3, \ldots, N_i$.

(3) Apply the Fast Fourier transform (FFT) to $y_i(t)$ to obtain $Y_i(\mathrm{j}\omega)$, then obtain the magnitude $|Y_i(\mathrm{j}\omega_0)|^2$ at frequency ω_0, and finally form a vector $\mathbf{YY} = [|Y_1(\mathrm{j}\omega_0)|^2, \ldots, |Y_{N_i}(\mathrm{j}\omega_0)|^2]^\mathrm{T}$.

(4) Obtain the following equation:

$$\begin{bmatrix} 1, & c_{11}, & c_{12}, & \ldots, & c_{1,K!} \\ 1, & c_{21}, & c_{21}, & \ldots, & c_{2,K!} \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ 1, & c_{N_i1}, & c_{N_i2}, & \ldots, & c_{N,K!} \end{bmatrix} \begin{bmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \vdots \\ \tilde{P}_{K!} \end{bmatrix} = \begin{bmatrix} |Y_1(j\omega_0)|^2 \\ |Y_2(j\omega_0)|^2 \\ \vdots \\ |Y_{N_i}(j\omega_0)|^2 \end{bmatrix}, \text{ i.e., } \boldsymbol{\psi}_{\mathbf{C}}\widetilde{\mathbf{P}}(j\omega_0) = \mathbf{YY}.$$

(5) Evaluate the function $\tilde{P}(j\omega_0)$ by using least squares

$$\tilde{P}(j\omega_0) = \left(\psi_C^{\mathrm{T}}\psi_C\right)^{-1}\psi_C^{\mathrm{T}}YY.$$

(6) Finally, the desired nonlinear controller parameters $\mathbf{C}^*$ for given Y^* at a specific frequency ω_0 can be determined according to

$$Y^* = \mathbf{C}^*\widetilde{\mathbf{P}}(j\omega_0).$$

The numerical method above is very effective for the implementation of the design of the proposed nonlinear controller parameters, which will be verified by a simulation study in Section 5.

Although there are some time domain methods which can address the nonlinear control problems based on Lyapunov stability theory such as the back-stepping technique and feedback linearization [18], etc., few results have been achieved for the design and analysis of a nonlinear feedback in the frequency domain to achieve a desired frequency domain performance. Based on the analytical relationship between system output spectrum and controller parameters defined by the OFRF, the analysis and design of nonlinear feedback can be conducted in the frequency domain. For a summary, a general procedure for this new method is given as follows:

(A) Determination of the structure of the nonlinear feedback function in Eq. (8).
This is to determine the value of M and choose the effective nonlinear controller parameters $C_{p0}(.)$ ($p = 2,3, \ldots, M$). This can follow the discussion in Section 3.2.

(B) Derivation of the region for the nonlinear feedback parameters in $C_{p0}(.)$ for $p = 2,3, \ldots, M$.
This is to ensure the stability of the nonlinear closed-loop system (10), which can be conducted by applying Proposition 3 to derive a stability condition for the closed-loop system in terms of the nonlinear controller parameters.

(C) Determination of the OFRF by using the numerical method and the optimal values for the nonlinear parameters.
This step includes two tasks. That is, (C1) determination of the detailed polynomial expression for the output spectrum according to Eq. (21) with respect to the specific nonlinear feedback (8) when the maximum nonlinearity order M is larger than 3, and (C2) determination of the desired value for each nonlinear controller parameter within the stability region to achieve the control objective (6) or (7) by using the numerical method provided above.

4. Simulation study

Consider a simple case of the model in Eqs. (1) and (2), which can be written as

$$\begin{cases} M\ddot{x} = -Kx - a_1\dot{x} + (\eta + u), \\ y = Kx + a_1\dot{x} - u. \end{cases}$$

This is the model of a vibration isolation system studied in Ref. [20] (Fig. 1), where $y(t)$ is the transmitted force from the disturbance $\eta(t)$ to the ground, K is a spring and a_1 is a damping.

Following the procedure in Section 3, a nonlinear feedback active controller $u(t)$ is designed and analysed for the suppression of output vibration in the frequency domain. It will be shown that a simple nonlinear feedback can bring much better improvement on the output performance, compared with a linear feedback.

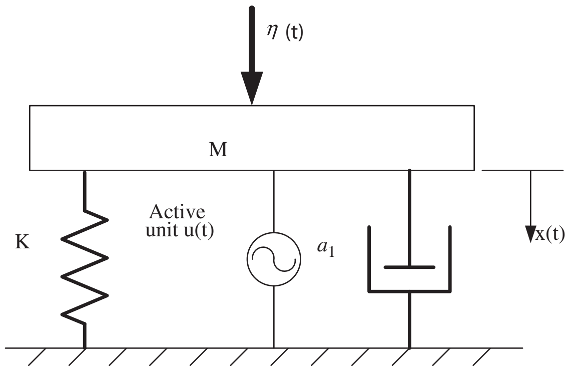

Fig. 1. A vibration isolation system.

The effect of the nonlinear feedback on system output performance is clearly demonstrated in the frequency domain by following the procedure above.

4.1. Determination of the structure of the nonlinear feedback controller

Considering the nonlinear feedback in Eq. (8), for this simple system, M is directly chosen to be 3, and all the other nonlinear controller parameters are chosen to be zero except $C_{30}(1\,1\,1) = a_3$ which represents a nonlinear damping. If $C_{30}(1\,1\,1) = a_3$ is not effective, more other nonlinear terms can be chosen. Later analysis will show that this choice is effective.

The nonlinear feedback control law now is

$$u = -a_3\dot{x}^3.$$

Then the closed-loop system is obtained as

$$\begin{cases} M\ddot{x} = -Kx - a_1\dot{x} - a_3\dot{x}^3 + \eta, \\ y = Kx + a_1\dot{x} + a_3\dot{x}^3. \end{cases} \tag{26a,b}$$

Note that system (26) is a very simple case of system (10), that is, $L = 2$, $\bar{C}_{10}(2) = M$, $\bar{C}_{10}(1) = a_1$, $\bar{C}_{10}(0) = K$, $\bar{C}_{30}(1\,1\,1) = a_3$, $\bar{C}_{01}(0) = -1$ and $\tilde{C}_{10}(1) = a_1$, $\tilde{C}_{10}(0) = K$, $\tilde{C}_{30}(1\,1\,1) = a_3$; all other parameters in model (10) are zero. Moreover, assume the disturbance input is $\eta(t) = F_d \sin(8.1t)$, which is a single tone function as a simple case of Eq. (12). Now the task for the nonlinear feedback controller design is to determine a_3 such that system (26) satisfies the control objective (7).

To verify the effectiveness of this nonlinear damping, the output spectrum should be computed up to the third order as discussed in Step(B). Noting that only $C_{30}(1\,1\,1) = a_3$ and all the other nonlinear parameters C_{p0} for $p>2$ are zero. According to Eqs. (18)–(20), the following parametric characteristics of the GFRFs can be obtained:

$$\mathrm{CE}(H_2^1(\cdot)) = \mathbf{C}_{20} \oplus \sum_{p=2}^{[(2+1)/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{2-p+1}^1(\cdot)) = \mathbf{C}_{20} = 0,$$

$$\mathrm{CE}(H_3^1(\cdot)) = \mathbf{C}_{30} \oplus \sum_{p=2}^{[(3+1)/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{3-p+1}^1(\cdot)) = \mathbf{C}_{30} = a_3,$$

$$\mathrm{CE}(H_4^1(\cdot)) = \mathbf{C}_{40} \oplus \sum_{p=2}^{[(4+1)/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{4-p+1}^1(\cdot)) = 0,$$

$$\mathrm{CE}(H_5^1(\cdot)) = \mathbf{C}_{50} \oplus \sum_{p=2}^{[(5+1)/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{5-p+1}^1(\cdot)) = \mathbf{C}_{30} \otimes \mathrm{CE}(H_3^1(\cdot)) = a_3^2, \ldots .$$

It is easy to check from Proposition 2 that

$$\mathrm{CE}(H_{2n+1}^{1}(\cdot)) = a_3^n \text{ for } n>0 \text{ and all other } \mathrm{CE}(H_i^1(\cdot)) = 0. \tag{27}$$

This shows that only $H_{2n+1}^{1}(\cdot)$ for $n>0$ are nonzero and all others are zero. Therefore, the output spectrum can be computed from Eqs. (13, 16) with only odd order GFRFs as

$$\begin{aligned} Y(j\omega) &= \sum_{n=1}^{N} \frac{1}{2^{2n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} H_{2n+1}^{2}(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_{2n+1}})F(\omega_{k_1})\cdots F(\omega_{k_{2n+1}}) \\ &= \frac{1}{2}H_1^2(\mathrm{j}\omega)F(\omega) + \frac{a_3}{8}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega} G_3^2(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2},\mathrm{j}\omega_{k_3})F(\omega_{k_1})F(\omega_{k_2})F(\omega_{k_3}) \\ &\quad + \frac{a_3^2}{32}\sum_{\omega_{k_1}+\cdots+\omega_{k_5}=\omega} G_5^2(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_5})F(\omega_{k_1})\cdots F(\omega_{k_5}) + \cdots, \\ &:= \bar{P}_0(\mathrm{j}\omega) + a_3\bar{P}_1(\mathrm{j}\omega) + a_3^2\bar{P}_2(\mathrm{j}\omega) + \cdots, \end{aligned} \tag{28a}$$

where

$$\begin{aligned} \bar{P}_0(\mathrm{j}\omega) &= \frac{1}{2}H_1^2(\mathrm{j}\omega)F(\omega) = \frac{-\mathrm{j}(a_1(\mathrm{j}\omega)+K)F_d}{2M(\mathrm{j}\omega)^2 + 2a_1(\mathrm{j}\omega)^1 + 2K}, \\ \bar{P}_1(\mathrm{j}\omega) &= -\frac{3}{8}MF_d^3\omega^5\left|H_1^1(\mathrm{j}\omega)\right|^2[H_1^1(\mathrm{j}\omega)]^2, \\ \bar{P}_2(\mathrm{j}\omega) &= -\frac{3j}{32}MF_d^5\left|\mathrm{j}\omega H_1^1(\mathrm{j}\omega)\right|^4[\mathrm{j}\omega H_1^1(\mathrm{j}\omega)]^2(\mathrm{j}\omega) \\ &\quad \times (\mathrm{j}3\omega H_1^1(\mathrm{j}3\omega) - \mathrm{j}3\omega H_1^1(-\mathrm{j}\omega) + \mathrm{j}6\omega H_1^1(\mathrm{j}\omega)). \end{aligned} \tag{28b}$$

Note that carrying out the computation above, the analytical relationship between the output spectrum and nonlinear parameter a_3 can be obtained explicitly for up to any high orders. It can be checked that $\mathrm{Re}(\bar{P}_0(\mathrm{j}\omega_0)\bar{P}_1(-\mathrm{j}\omega_0)) = 0.5(\bar{P}_0(\mathrm{j}\omega_0)\bar{P}_1(-\mathrm{j}\omega_0) + \bar{P}_0(-\mathrm{j}\omega_0)\bar{P}_1(\mathrm{j}\omega_0)) = -31.132<0$ when $a_3>0$, $\omega_0 = 8.1$ rad/s and other system parameters as given in the simulation studies. Hence, the nonlinear control parameter a_3 is effective. If there are other nonlinear controller parameters, the same method can be used to check the effectiveness as discussed in Step(B). Only the effective nonlinear terms are used in the controller.

4.2. Derivation of the stability region for the parameter a_3

According to Proposition 3, the following result can be obtained.

Proposition 4. *Consider the closed-loop system* (26), *and suppose the exogenous disturbance input satisfies* $\|\eta(t)\| \leqslant F_d$. *The system is asymptotically stable to a ball* $B_{F_d\sqrt{\lambda_{\min}(Q)^{-1}\varepsilon}}(\mathbf{X})$, *if* $a_3>0$ *and additionally there exist* $\mathbf{P} = \mathbf{P}^{\mathrm{T}}>0$, $\beta>0$ *and* $\varepsilon>0$ *such that*

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{A} - \varepsilon^{-1}\mathbf{P}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P} & -\beta\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} + \mathbf{P}\mathbf{B} - \beta\mathbf{P}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} \\ * & +2\beta\mathbf{C}\mathbf{B} - \varepsilon^{-1}\beta^2\mathbf{C}\mathbf{E}\mathbf{E}^{\mathrm{T}}C^{\mathrm{T}} \end{bmatrix} > 0.$$

Moreover, the closed-loop system (26) *without a disturbance input is global asymptotically stable if the above inequality holds with* $\mathbf{E} = 0$. *Here*,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{a_1}{M} \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0, & \dfrac{1}{M} \end{bmatrix}^{\mathrm{T}},\ \mathbf{C} = [0, 1],\ \mathbf{E} = \begin{bmatrix} 0, & \dfrac{1}{M} \end{bmatrix}^{\mathrm{T}}.$$

Proof. See Appendix A. □

It is noted that the inequality in Proposition 4 has no relation with a_3, the left part of the inequality is determined by the linear part of system (26), and the whole inequality can be checked by using the linear matrix inequalities technique [21]. This also implies that the value of a_3 has no effect on the stability of the system if the inequality is satisfied. Hence, the nonlinear controller parameter a_3 is now only restricted to the region $[0, \infty)$, provided that the linear system satisfies the inequality condition.

4.3. Derivation of the OFRF and determination of the desired value of the nonlinear parameter a_3

Using Eq. (27), the parametric characteristics of the output spectrum of nonlinear system (26) can be obtained as

$$\mathrm{CE}(Y(\mathrm{j}\omega)) = \mathrm{CE}(H_1^1(\cdot)) \oplus \mathrm{CE}(H_2^1(\cdot)) \oplus \cdots \oplus \mathrm{CE}(H_N^1(\cdot)) = \begin{bmatrix} 1 & a_3 & a_3^2 & \cdots & a_3^Z \end{bmatrix},$$

where $Z = \lfloor (N-1)/2 \rfloor$. Therefore, the system output spectrum can be written as a polynomial expression as

$$Y(\mathrm{j}\omega) = \bar{P}_0(\mathrm{j}\omega) + a_3\bar{P}_1(\mathrm{j}\omega) + a_3^2\bar{P}_2(\mathrm{j}\omega) + \cdots + a_3^Z\bar{P}_Z(\mathrm{j}\omega).$$

Hence,

$$\begin{aligned} Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega) = \left|Y(\mathrm{j}\omega)\right|^2 &= \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 + a_3\left(\bar{P}_0(\mathrm{j}\omega)\bar{P}_1(-\mathrm{j}\omega) + \bar{P}_0(-\mathrm{j}\omega)\bar{P}_1(\mathrm{j}\omega)\right) + a_3^2(\left|\bar{P}_1(\mathrm{j}\omega)\right|^2 \\ &+ \bar{P}_0(\mathrm{j}\omega)\bar{P}_2(-\mathrm{j}\omega) + \bar{P}_0(-\mathrm{j}\omega)\bar{P}_2(\mathrm{j}\omega)) + \cdots . \end{aligned} \tag{28c}$$

Clearly, $|Y(\mathrm{j}\omega)|^2$ is also a polynomial function of a_3. Given the magnitude of a desired output frequency response Y^* at any frequency ω_0, a_3 can be solved from Eq. (28c) provided that $|Y(\mathrm{j}\omega)|$ can be approximated by a polynomial expression of a finite order. In order to determine a desired value for a_3 to achieve the control objective (7), the numerical method proposed in Section 3.4 is used. Since Eq. (28c) is a polynomial function of a_3, $|Y(\mathrm{j}\omega)|^2$ can be directly approximated by a polynomial function of a_3 without computation of higher order GFRFs as follows:

$$Y(\mathrm{j}\omega)Y(-\mathrm{j}\omega) = \left|Y(\mathrm{j}\omega)\right|^2 \approx a_3^{2Z}\tilde{P}_{2Z} + \cdots + a_3^n\tilde{P}_n + a_3^{n-1}\tilde{P}_{n-1} + \cdots + a_3\tilde{P}_1 + \left|\bar{P}_0(\mathrm{j}\omega)\right|^2, \tag{29a}$$

where $|Y(\mathrm{j}\omega)|^2$ can be obtained through FFT of the data from simulations or experiments. Given $2Z$ different values of a_3, i.e., a_{31}, a_{32}, ..., $a_{3,2Z}$, it can be further written as (for each values of a_3)

$$\left|Y(\mathrm{j}\omega)_i\right|^2 \approx a_{3i}^{2Z}\tilde{P}_{2Z} + \cdots a_{3i}^n\tilde{P}_n + a_{3i}^{n-1}\tilde{P}_{n-1} + \cdots + a_{3i}\tilde{P}_1 + \left|\bar{P}_0(\mathrm{j}\omega)\right|^2$$

for $i = 1, 2, \ldots, 2Z$, i.e.,

$$\begin{bmatrix} a_{31} & a_{31}^2 & a_{31}^3 & \cdots & a_{31}^{2Z} \\ a_{32} & a_{32}^2 & a_{32}^3 & \cdots & a_{32}^{2Z} \\ & & & \ddots & \vdots \\ a_{3,2Z} & a_{3,2Z}^2 & a_{3,2Z}^3 & \cdots & a_{3,2Z}^{2Z} \end{bmatrix} \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \vdots \\ \tilde{P}_{2Z} \end{bmatrix} = \begin{bmatrix} \left|Y(\mathrm{j}\omega)_1\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \\ \left|Y(\mathrm{j}\omega)_2\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \\ \vdots \\ \left|Y(\mathrm{j}\omega)_{2Z}\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \end{bmatrix}.$$

Then $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_{2Z}$ are obtained as

$$\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \vdots \\ \tilde{P}_{2Z} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{31}^2 & a_{31}^3 & \cdots & a_{31}^{2Z} \\ a_{32} & a_{32}^2 & a_{32}^3 & \cdots & a_{32}^{2Z} \\ & & & \ddots & \vdots \\ a_{3,2Z} & a_{3,2Z}^2 & a_{3,2Z}^3 & \cdots & a_{3,2Z}^{2Z} \end{bmatrix}^{-1} \begin{bmatrix} \left|Y(\mathrm{j}\omega)_1\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \\ \left|Y(\mathrm{j}\omega)_2\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \\ \vdots \\ \left|Y(\mathrm{j}\omega)_{2Z}\right|^2 - \left|\bar{P}_0(\mathrm{j}\omega)\right|^2 \end{bmatrix}. \tag{29b}$$

Consequently, Eq. (29a) is obtained. Using this method, a polynomial expression of $|Y(\mathrm{j}\omega)|^2$ in any order can be achieved. Given a desired output frequency response Y^* at a frequency ω_0, a_3 can be solved from Eq. (29a) to implement the design. Note that roots of Eq. (29a) are multiple. According to Proposition 4, the solution a_3 should be a nonnegative real number.

4.5. Simulation results

In the simulation, the parameters of system (26) are $K = 16{,}000$ N/m, $a_1 = 296$ N s/m, $M = 240$ kg. The resonant frequency of the system is $\omega_0 = 8.1$ rad/s. In order to show the effectiveness and advantage of the nonlinear feedback controller $u = -a_3\dot{x}^3$, a linear controller $u = -a_2\dot{x}$ will be used for comparison.

Firstly, let $F_d = 100$ N. We need to obtain the polynomial function (29a). In order to have a larger working region of a_3, let $Z = 6$ in Eq. (29a), and $a_3 = 500$, 1000, 2000, 4000, 6000, 8000, 10,000, 12,000, 14,000, 16,000, 18,000, and 20,000. Under these different values of a_3, the output frequency response of the system was obtained and the corresponding output spectrum was determined via FFT operations. Then $\tilde{P}_n(\mathrm{j}\omega)$ for $n = 1, \ldots, 12$ were obtained according to Eq. (29b), which are summarized partly in Table 1. For comparisons, the corresponding theoretical results were also computed from Eq. (28a–c) and are given partly in Table 1. From Table 1, it can be seen that there is a good match between the data analysis results and the theoretical computations although there are some errors. This result shows that the theoretical computation results are basically consistent with the results from the simulation analyses. It can also be seen from the numerical analysis results in Table 1 that Eq. (29a) is in fact an alternative series in this case.

Fig. 2 shows the results of the system output spectrum under different values of the nonlinear control parameter a_3 and provides a comparison between theoretical computations using polynomial expression (28c) up to the third order and the numerical results using the polynomial expression (29a) up to the 12th order. This result demonstrates the analytical relationship between the nonlinear control parameter and the system output spectrum, and shows that the theoretical results have a good match with the numerical results when a_3 is small since only up to the third-order GFRF are used in the theoretical computations. Hence, with an increase of a_3, the numerical method has to be used in order to give correct results. Moreover, it should be noted that the magnitude of the system output spectrum decreases with the increase of a_3. This verifies that the nonlinear control parameter a_3 is effective for the control problem.

Without a control input, the system output frequency spectrum is as shown in Fig. 3(b), where $Y(\mathrm{j}\omega)\big|_{\omega_0} = 335.71$. Note that the output response spectrum shown in the figures of this paper is $2|Y|$ not $|Y|$, that is also applied on the plot of the output spectrum using the theoretical computation. This is because $2|Y|$ represents the physical magnitude of the system output at the frequency ω_0. If the desired output frequency spectrum is set to be $Y^* = 180$, then the calculation according to Eq. (29a,b) and Proposition 4 yields $a_3 = 11{,}869$. The output frequency spectrum under the nonlinear feedback control is shown in Fig. 3(a), where $Y(\mathrm{j}\omega)\big|_{\omega_0} = 180.08$, and hence the result matches the desired result quite well. The system outputs in the time domain before and after nonlinear feedback control are given in Fig. 4. It can be seen that the system steady-state performance is considerably improved when the nonlinear controller is used.

In order to further demonstrate the advantage of the nonlinear feedback, consider a linear damping controller $u = -275\dot{x}$. Under this linear control input, the system output frequency response as shown in Fig. 5 is similar to that achieved with the nonlinear controller. However, when F_d is increased to 200 N, the output frequency response is quite different under the two controllers. The nonlinear feedback controller results in a much smaller magnitude of output frequency response at frequency ω_0, referring to Fig. 6. Fig. 7 shows the results of the system outputs in the time domain under the two different control inputs, indicating

Table 1
Comparison between simulation and theoretical results

Simulation results		Theoretical results	
$\lvert\bar{P}_0(\mathrm{j}\omega)\rvert^2$	1.1270e+05	$\lvert\bar{P}_0(\mathrm{j}\omega)\rvert^2$	1.1257e+05
$\tilde{P}_1$	−58.9652	$\bar{P}_0(\mathrm{j}\omega)\bar{P}_1(-\mathrm{j}\omega) + \bar{P}_0(-\mathrm{j}\omega)\bar{P}_1(\mathrm{j}\omega)$	−62.2641
$\tilde{P}_2$	0.0423	$\lvert\bar{P}_1(\mathrm{j}\omega)\rvert^2 + \bar{P}_0(\mathrm{j}\omega)\bar{P}_2(-\mathrm{j}\omega) + \bar{P}_0(-\mathrm{j}\omega)\bar{P}_2(\mathrm{j}\omega)$	0.0615
$\tilde{P}_3$	−2.3762e−005	–	–
$\tilde{P}_4$	9.1382e−009	–	–
$\tilde{P}_5$	−2.3593e−012	–	–
⋮	⋮	⋮	⋮

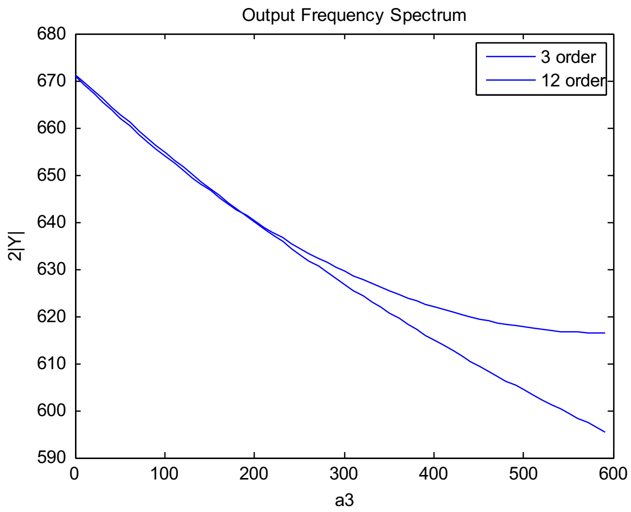

Fig. 2. Analytical relationship between the system output spectrum and the control parameter a_3.

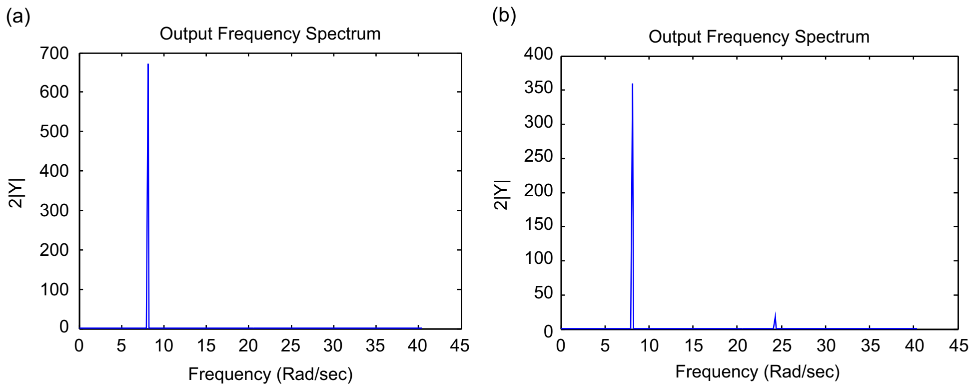

Fig. 3. Output spectrum: (a) without a feedback control and (b) with the designed nonlinear feedback.

the nonlinear controller has a much better result than the linear controller. When the input frequency ω_0 is increased to be 15 rad/s, the same conclusions can be reached for the two controllers, referring to Fig. 8. When the input frequency is decreased to be 5 rad/s, the output spectrums under the two controllers are similar (see Fig. 9). On the otherhand, although increase of the liner damping can also achieve better output performance at the driving frequency, this will degrade the output performance at high frequencies as known in literature (Fig. 10). However, the nonlinear damping has no obviously such a limitation (Fig. 11).

The results demonstrate that a simple nonlinear feedback to realize a cubic nonlinear damping can achieve better performances than a linear damping control for vibration suppression both in low and high frequencies. The frequency domain method proposed in this study provides an effective approach to the analysis and design of the nonlinear feedback. Although only a simple case with only one nonlinear term is studied in this simulation, much more complicated cases with multiple nonlinear parameters can also be analysed and designed straightforward by following the same method. It should be noted that there may be some other methods in the literature which can be used to realize the same control purpose of this study, however, the advantage of this method is that it can directly relate the nonlinear controller parameters to system output frequency response and therefore the nonlinear controller or structural parameters can be analysed and

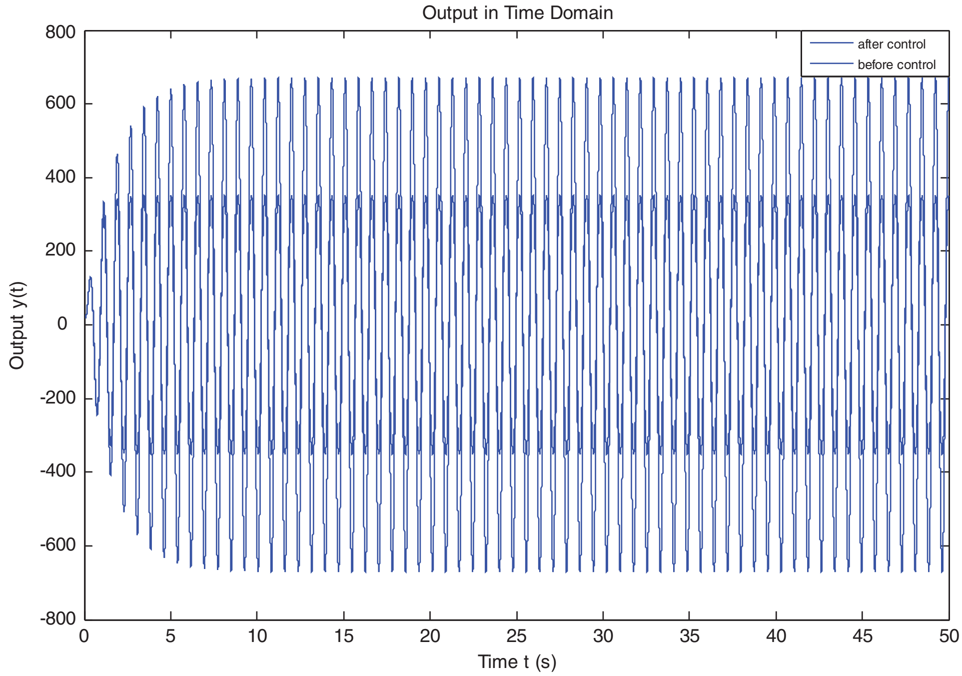

Fig. 4. System output in time domain: before and after control.

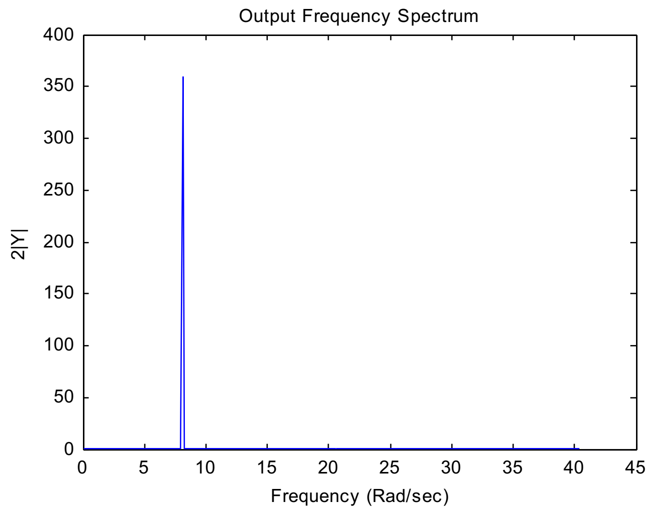

Fig. 5. Output spectrum with the linear feedback control.

designed in the frequency domain, which is a more understandable way in engineering practice. Furthermore, the designed controller, for instance the nonlinear damping designed in the example study above, may also be realized by a passive unite, and the analysis by using this method can be performed directly for a physical characteristics of a structural unite in a system. This will have great significance in practical applications.

5. Conclusions

A frequency domain approach to the analysis and design of nonlinear feedback to suppress periodic disturbance for SISO plants is studied and theoretical results associated with this subject are investigated.

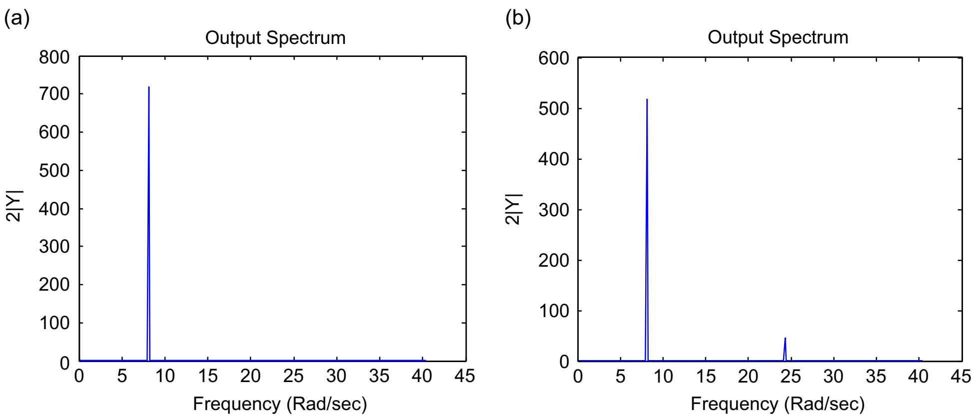

Fig. 6. Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control, when F_d is increased to $F_d = 200$ ($a_2 = 275$, $a_3 = 11{,}869$).

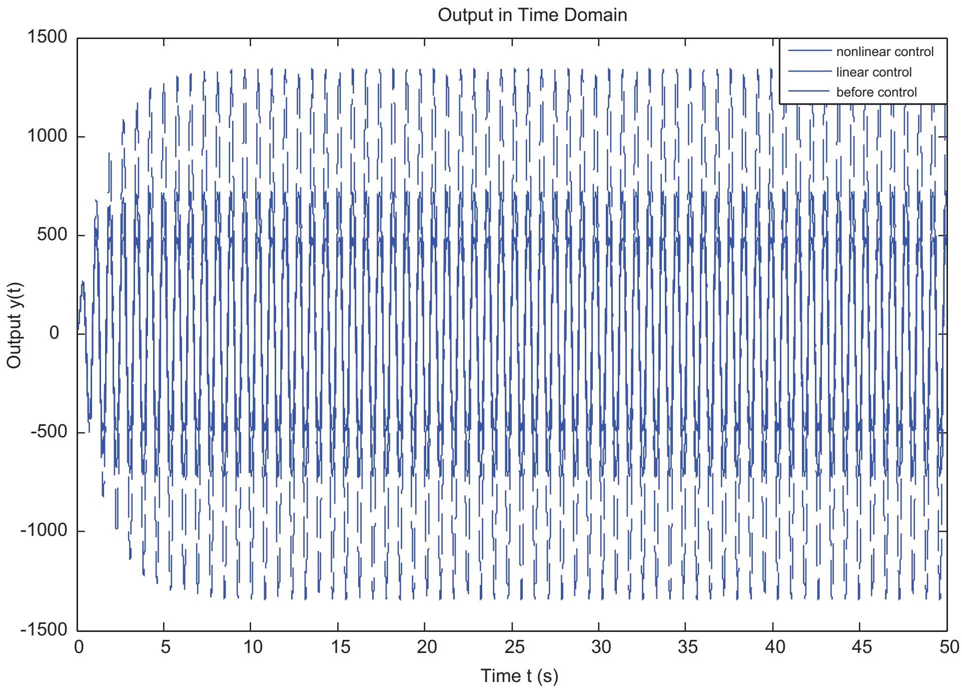

Fig. 7. The system outputs in time domain under different control inputs ($F_d = 200$).

Although there are already some time domain methods, which can address the nonlinear control problems based on Lyapunov stability theory, few results have been achieved for the design and analysis of a nonlinear feedback in the frequency domain to achieve a desired frequency domain performance. Based on the analytical relationship between system output spectrum and controller parameters, this paper provides, for the first time, a systematic frequency domain approach to exploiting the potential advantage of nonlinearities to achieve a desired output frequency domain performance for the analysis and design of vibration systems. Compared with other existing methods for the same purposes, the method in this paper can directly relate the nonlinear parameters of interest to the system output frequency response and the designed controller may also be realized by a passive unite in practices. Although the results in this paper are developed for the problem of

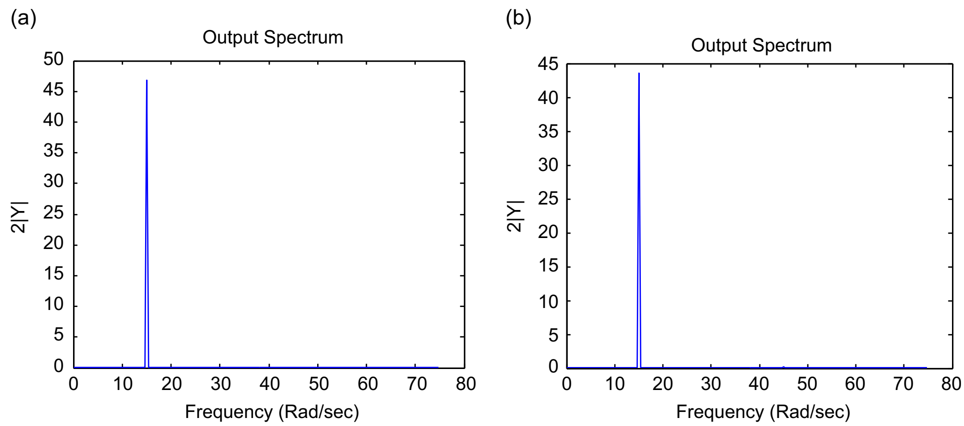

Fig. 8. Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control, when $\omega_0 = 15\,\text{rad/s}$, $F_d = 100$, $a_2 = 275$, $a_3 = 11{,}869$.

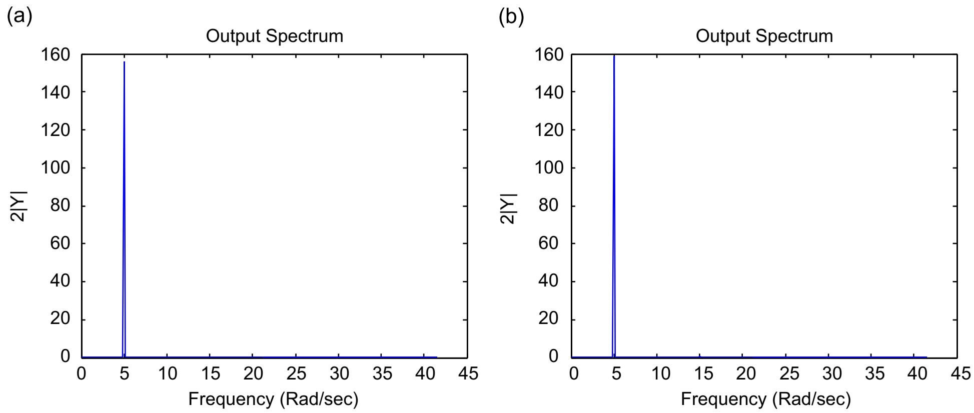

Fig. 9. Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control, when $\omega_0 = 5\,\text{rad/s}$, $F_d = 100$, $a_2 = 275$, $a_3 = 11{,}869$.

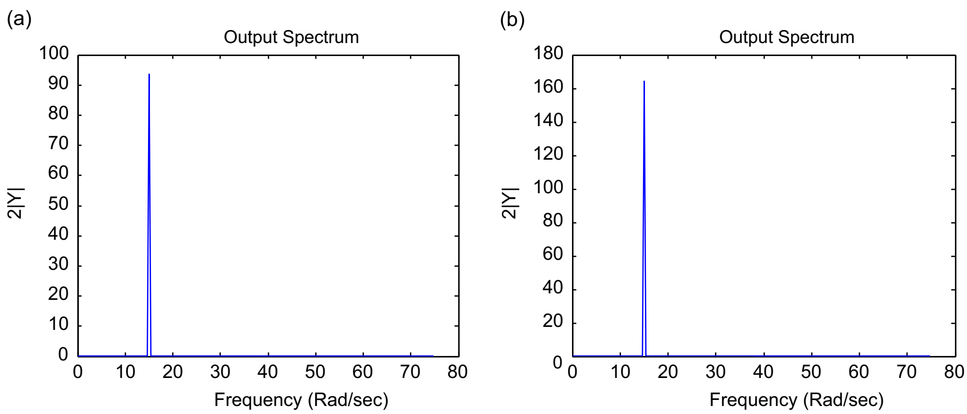

Fig. 10. Output spectrum with the linear feedback control when (a) $a_2 = 275$ and (b) $a_2 = 2750$ ($\omega_0 = 15\,\text{rad/s}$, $F_d = 200$).

periodic disturbance suppression for SISO linear plants, the idea can be extended to a more general case (i.e., nonlinear controlled plants) and to address more complicated control problems. Future studies will focus on these related issues.

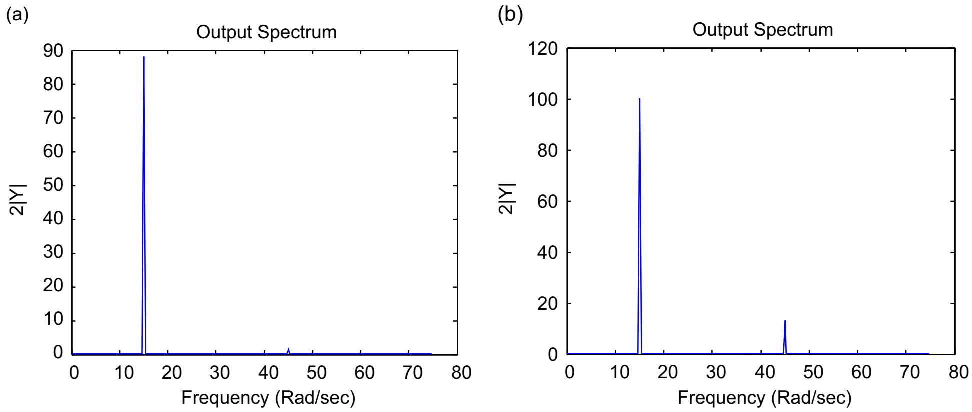

Fig. 11. Output spectrum with the nonlinear feedback control when (a) $a_3 = 11{,}869$ and (b) $a_3 = 1{,}18{,}690$ ($\omega_0 = 15\,\text{rad/s}$, $F_d = 200$).

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Appendix A. Proofs

Proof of Proposition 1. Regard $x(t)$ and $y(t)$ as two outputs of system (10), i.e., $y_1(t) = x(t)$, $y_2(t) = y(t)$, and the exogenous disturbance $\eta(t)$ as the input, then derive the GFRFs $H_n^j(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n})$ (for $j = 1{,}2$ and $n = 1{,}2{,}3,\ldots$) for system (10). Consider Eq. (10a), which is consistent with the model studied in Ref. [9]. Hence, the nth GFRF of Eq. (10a), denoted by $H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$, can be obtained by directly applying the results in Ref. [9] as follows:

$$H_n^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = \frac{-1}{\sum_{l_1=0}^{L}\bar{C}_{10}(l_1)(\mathrm{j}\omega_1+\cdots+\mathrm{j}\omega_n)^{l_1}}\left(\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\bar{C}_{p0}(l_1\cdots l_p)H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)\right.$$
$$\left.-\sum_{l_1\cdots l_n=0}^{L}\bar{C}_{0n}(l_1\cdots l_n)(\mathrm{j}\omega_1)^{l_1}\cdots(\mathrm{j}\omega_n)^{l_n}\right), \tag{A.1}$$

where $\bar{C}_{01}(0) = e$, all other $\bar{C}_{0n}(\cdot) = 0$; $H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ and $H_{n,1}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)$ are Eqs. (16b) and (16c), respectively. Note that $\bar{C}_{01}(0) = e$ and all the other $\bar{C}_{0n}(\cdot) = 0$, the first-order GFRF (linear frequency-response function) is

$$H_1^1(\mathrm{j}\omega_1) = \frac{\sum_{l_1=0}^{L}\bar{C}_{01}(l_1)(\mathrm{j}\omega_1)^{l_1}}{\sum_{l_1=0}^{L}\bar{C}_{10}(l_1)(\mathrm{j}\omega_1)^{l_1}} = \frac{e}{\sum_{l_1=0}^{L}\bar{C}_{10}(l_1)(\mathrm{j}\omega_1)^{l_1}}. \tag{A.2}$$

Using Eq. (A.2), Eq. (A.1) can be rewritten into Eq. (16d). Consider Eq. (10b), which has two outputs, i.e., one pure output nonlinearities in terms of $D^i x(t)$ and one linear pure output term $y(t)$. Note that there is no nonlinearities in terms of $y(t)$. Hence, the nth-order GFRF of the output $y(t)$ is completely dependent on the nth-order GFRF of the first "output" $x(t)$. Following the probing method [22] and also referring to the discussions in Ref. [9], it can be obtained as

$$H_n^2(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = \sum_{p=1}^{n}\sum_{l_1\cdots l_p=0}^{L}\tilde{C}_{p0}(l_1\cdots l_p)H_{n,p}^1(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n).$$

Note that here $p=1$ represents the linear effect from $x(t)$, which is different from Eq. (A.1) where p is counted from 2. This is Eq. (16a). This completes the proof. □

Proof of Proposition 3. To prove Proposition 3, the following lemmas are needed.

Lemma 3. *Consider two positive, scalar and continuous process in time t, $x(t)$ and $y(t)$ satisfying $y(t)\leqslant\alpha(x(t))$ (for $t\geqslant 0$), where α is a K-function. If $x(t)$ is asymptotically stable to a ball $B_\rho(x)$, then $y(t)$ is asymptotically stable to a ball $B_{\alpha(2\rho)}(y)$.*

Proof. There exists a *KL*-function β, such that function $x(t)$ (for $t\geqslant 0$) satisfies $x(t)\leqslant\beta(x(0),t)+\rho$, $\forall t>0$. Therefore,
$y(t)\leqslant\alpha(x(t))=\alpha(\beta(x(0),t)+\rho)\leqslant\alpha(\max(2\beta(x(0),t),2\rho))=\max(\alpha(2\beta(x(0),t)),\alpha(2\rho))\leqslant\alpha(2\beta(x(0),t))+\alpha(2\rho)$. Note that $\alpha(2\beta(x(0),t))$ is still a *KL*-function of $x(0)$ and t, thus the lemma is concluded. □

From Lemma 3, if there exists a *K*-function σ such that the output function $h(\mathbf{X})$ of a nonlinear system satisfies $\|h(\mathbf{X})\|\leqslant\sigma(\|\mathbf{X}\|)$, then the system output is asymptotically stable to a ball if the system is asymptotically stable to a ball.

Lemma 4. *Consider a scalar differential inequality $\dot{y}(t)\leqslant-\alpha(y(t))+\gamma$, where α is a K-function and γ is a constant and $y(t)$ satisfies Lipschitz condition. Then there exists KL-function β such that*

$$y(t)\leqslant\beta(\left|y(t_0)-\alpha^{-1}(\gamma)\right|,t)+\alpha^{-1}(\gamma).$$

Proof. Consider the differential equation $\dot{y}(t)=-\alpha(y(t))$. From Lemma 10.1.2 in Ref. [18] it is known that, there is a *KL*-function β such that $y(t)=\beta(y(t_0),t)$. Similarly, considering the differential equation $\dot{y}(t)=-\alpha(y(t))+\gamma$, then $y(t)=\mathrm{sign}(y(t_0)-\alpha^{-1}(\gamma))\ \beta(|y(t_0)-\alpha^{-1}(\gamma)|,t)+\alpha^{-1}(\gamma)$. Thus from the comparison principle and the differential inequality $\dot{y}(t)\leqslant-\alpha(y(t))+\gamma$, the lemma follows. □

Then to prove Proposition 3, it follows from Eq. (25) that

$$\dot{V}(\mathbf{X}(t))\leqslant-\alpha(\|\mathbf{X}\|)+\gamma(\|\eta\|_\infty). \tag{A.3}$$

Noting $V(\mathbf{X})\leqslant\beta_2(\|\mathbf{X}\|)$, we have $\|\mathbf{X}\|\geqslant\beta_2^{-1}(V(\mathbf{X}))$. Substituting this inequality into Eq. (A.3), we have

$$\dot{V}(\mathbf{X}(t))\leqslant-\alpha(\beta_2^{-1}(V(\mathbf{X})))+\gamma(\|\eta\|_\infty).$$

From Lemma 4, it follows that, there exist a *KL*-function β, such that

$$V(\mathbf{X}(t))\leqslant\beta(V_0,t)+\beta_2^{-1}\alpha^{-1}\gamma(\|\eta\|_\infty), \tag{A.4}$$

where $V_0=|V(\mathbf{X}(t_0))-\beta_2^{-1}\alpha^{-1}\gamma(\|\eta\|_\infty)|$. From Eq. (A.4), $V(\mathbf{X}(t))$ is asymptotically stable to the ball $B_{\beta_2^{-1}\alpha^{-1}\gamma(\|\eta\|_\infty)}(V)$. Noting $\beta_1(\|\mathbf{X}\|)\leqslant V(\mathbf{X})$, we have $\|\mathbf{X}\|\leqslant\beta_1(V(\mathbf{X}))$. From Lemma 3, $\mathbf{X}(t)$ is asymptotically stable to the ball $B_\rho(\mathbf{X})$. Furthermore, since Assumption 1 holds, from Lemma 3, $y(t)$, is asymptotically stable to the ball $B_{o(2\rho)}(y)$. This completes the proof of sufficiency. The proof of the necessity of the proposition can follow a similar method as demonstrated in the appendix of Ref. [23]. The proof completes. □

Proof of Proposition 4. The state-space equation of system (26a) can be written as $\dot{\mathbf{X}}=\mathbf{AX}-\mathbf{B}\phi+\mathbf{E}\eta$, where $\mathbf{X}=[x,\dot{x}]^{\mathrm{T}}$, $\phi=a_3\sigma^3$, $\sigma=\mathbf{CX}$. Choose a Lyapunov candidate as

$$V=\mathbf{X}^{\mathrm{T}}\mathbf{PX}+\frac{\alpha}{2}\sigma^4, \tag{A.5}$$

where $\alpha>0$. Eq. (A.5) further follows:

$$\begin{aligned}\dot{V}&=\mathbf{X}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{X}}+\dot{\mathbf{X}}^{\mathrm{T}}\mathbf{PX}+2\alpha\sigma^3\mathbf{C}\dot{\mathbf{X}}=\mathbf{X}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{PA})\mathbf{X}-2\mathbf{X}^{\mathrm{T}}\mathbf{PB}\phi+2\mathbf{X}^{\mathrm{T}}\mathbf{PE}\eta\\&\quad+\frac{2\alpha}{a_3}\phi\mathbf{C}(\mathbf{AX}-\mathbf{B}\phi+\mathbf{E}\eta)=\mathbf{X}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{PA})\mathbf{X}-2\mathbf{X}^{\mathrm{T}}\mathbf{PB}\phi+\frac{2\alpha}{a_3}\phi\mathbf{CAX}-\frac{2\alpha}{a_3}\phi\mathbf{CB}\phi+2\mathbf{X}^{\mathrm{T}}\mathbf{PE}\eta\\&\quad+\frac{2\alpha}{a_3}\phi\mathbf{CE}\eta.\end{aligned} \tag{A.6}$$

Let $\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \phi \end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} \mathbf{PE} \\ \frac{\alpha}{a_3}\mathbf{CE} \end{bmatrix}$, and $\beta = \alpha/a_3$ then Eq. (A.6) follows:

$$\begin{aligned} \dot{V} &= \mathbf{Z}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{PB} \\ * & -2\beta\mathbf{CB} \end{bmatrix} \mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\mathbf{T}\eta \leqslant \mathbf{Z}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{PB} \\ * & -2\beta\mathbf{CB} \end{bmatrix} \mathbf{Z} \\ &\quad + \varepsilon^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{T}\mathbf{T}^{\mathrm{T}}\mathbf{Z} + \varepsilon\eta^{\mathrm{T}}\eta = \mathbf{Z}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} - \mathbf{PB} \\ * & -2\beta\mathbf{CB} \end{bmatrix} + \varepsilon^{-1}\mathbf{T}\mathbf{T}^{\mathrm{T}} \right) \mathbf{Z} + \varepsilon\eta^2 \\ &= -\mathbf{Z}^{\mathrm{T}}\mathbf{QZ} + \varepsilon\eta^2. \end{aligned}$$

Note that, in the inequality above, the following inequality is used:

$$2\mathbf{Z}^{\mathrm{T}}\mathbf{T}\eta \leqslant \varepsilon^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{T}\mathbf{T}^{\mathrm{T}}\mathbf{Z} + \varepsilon\eta^{\mathrm{T}}\eta \quad \text{for any } \varepsilon > 0.$$

If $\mathbf{Q} = \mathbf{Q}^{\mathrm{T}} > 0$, then $\mathbf{Z}^{\mathrm{T}}\mathbf{QZ} \geqslant \lambda_{\min}(\mathbf{Q})\|\mathbf{X}\|^2$ is a K-function of $\|\mathbf{X}\|$. Hence, according to Proposition 3, the system is asymptotically stable to a ball $B_\rho(\mathbf{X})$ with $\rho = \sqrt{\lambda_{\min}(\mathbf{Q})^{-1}\varepsilon\sup(\|\eta\|^2)} = F_d\sqrt{\lambda_{\min}(\mathbf{Q})^{-1}\varepsilon}$. Additionally, when there is no exogenous disturbance input, and if $\mathbf{Q} = \mathbf{Q}^{\mathrm{T}} > 0$ holds with $\mathbf{E} = 0$, then it is obvious that the system without a disturbance input is globally asymptotically stable. This completes the proof. □

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